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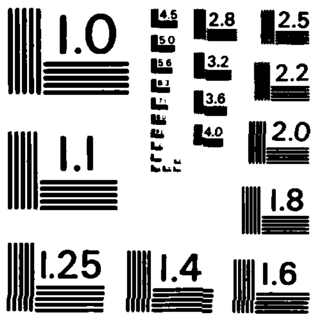

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This paper presents a simple way of classifying higher order differential equations based on the requirements of the Painlevé property, i.e., the presence of no movable critical points. The fundamental building blocks for such equations may be generated by strongly self-dominant differential equations of the type $(\partial/\partial x)^n u=(\partial/\partial x^m)[u^{(m-n+p)/p}]$ in which m and n are positive integers and p is a negative integer. Such differential equations having both a constant degree d and a constant value of the difference n-m form a Painlevé chain; however, only three of the many possible Painlevé chains can have the Painlevé property. Among the three (cont.)

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Abstract

This paper presents a simple way of classifying higher order differential equations based on the requirements of the Painlevé property, i.e., the presence of no movable critical points. The fundamental building blocks for such equations may be generated by strongly self-dominant differential equations of the type $(\partial/\partial x)^n u = (\partial/\partial x^m)[u^{(m-n+p)/p}]$ in which m and n are positive integers and p is a negative integer. Such differential equations having both a constant degree d and a constant value of the difference *n-m form a* <u>Painlevé chain</u>; however, only three of the many possible Painlevé chains can have the Painlevé property. Among the three Painlevé chains which can have the Painlevé property, one contains the Burgers' equation; another contains the dominant terms of the first Painlevé transcendent, the isospectral Korteweg-de Vries equation, and the isospectral Boussinesq equation; and the third contains the dominant terms of the second Painlevé transcendent and the isospectral modified (cubic) Korteweg-de Vries equation. Differential equations of the same order and having the same value of the quotient (n-m)/(d-1) can be mixed to generate a new hybrid differential equation. In such cases a hybrid can have the Painlevé property even if only one of its components has the Painlevé property. Such hybridization processes can be used to generate the various fifth-order evolution equations of interest, namely the Caudrey-Dodd-Gibbon, Kuperschmidt, and Morris equations.

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TECHNICAL REPORT NO. 14

Systematics of Strongly Self-dominant Higher Order Differential Equations Based on the Painlevé Analysis of their Singularities

by

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1. INTRODUCTION

In recent years theoretical physicists have been very interested in a certain class of non-linear partial differential equations known as evolution equations.[1,2] This interest has arisen from the realization that these equations possess a special type of elementary solution which takes the form of localized disturbances which act somewhat like particles and are therefore known as solitons.

Solution of these evolution equations involves the so-called inverse scattering transform.[1,2,3] In this connection it has been noted[4,5,6] that there is a relation between non-linear partial differential equations solvable by an inverse scattering transform and non-linear ordinary differential equations (ODE) without movable critical points; such ODE's are said to have the Painlevé property. An algorithm has been developed[5,6] for the determination whether a given system of differential equations has the Painlevé property.

In applying the algorithm to test systems of differential equations for the Painlevé property, the question naturally arises to what extent useful information can be obtained on the properties of a given differential equation by simple inspection of the equation without extensive calculation. This paper explores this question and provides a simple approach for identifying higher order differential equations possessing necessary conditions to have the Painlevé property. More extensive tests are then required in order to determine whether these necessary conditions are sufficient for specific equations to have the Painlevé property.

II. DOMINANCE CLASSES AND FAMILIES

In this work we are interested in evolution equations of the form

$$u_t + f(x,u,u_{1x},\dots,u_{nx})=0 \tag{1}$$

in which

$$u_t = \frac{\partial u}{\partial t} \text{ and } u_{jx} = (\frac{\partial}{\partial x})^j u \qquad j=0,1,\dots,n \tag{2}$$

In these equations u may be regarded as an amplitude, x as a distance, and t as time. Of particular interest are time-independent solutions where $u_t=0$ and therefore

$$f(x,u,u_{1x},\dots,u_{nx})=0 \tag{3}$$

Let us adjust the distance scale x so that $x=x^*$ is a critical point. The dominant behavior of solutions of equation 3 in the neighborhood of the critical point $x=x^*$ can be expressed as the following series:

$$u=a(x - x^*)^p \text{ as } x \to x^* \tag{4}$$

Substitution of equation 4 into equation 3 shows that for certain values of the exponent p, two or more terms may balance (possibly depending upon a) and the rest can be ignored as $x \to x^*$. For each choice of p the terms which can balance are called the dominant terms. Modification of equation 3 by deletion of all non-dominant terms in general leads to a new simpler equation called the dominant truncation of equation 3. All equations giving the same dominant truncation may be considered as forming a dominance class. All dominance classes which have identical dominant truncations except for multiplicative constants may be regarded as forming a dominance family. A self-dominant equation is one in which all of its terms are dominant and is therefore identical to its dominant truncation.

Painlevé[7] has identified 50 canonical forms of second order differential equations which lack movable critical points and which therefore are related to non-linear differential equations solvable by inverse scattering transforms. The methods and ideas to be used in this paper can first be illustrated with the simplest irreducible Painlevé equation, namely

$$u_{xx} = 6u^2 + x \tag{5}$$

Expressing u by equation 4 gives

$$ap(p-1)(x - x^*)^{p-2} = 6a^2(x - x^*)^{2p} + x \tag{6}$$

Balancing the $(x - x^*)^{p-2}$ term on the left with the $(x - x^*)^{2p}$ term on the right gives $p - 2 = 2p$ or $p = -2$. The x term on the right of equations 5 and 6 is not involved in the balancing. Such terms are called recessive terms and are those dropped from the differential equation to form its dominant truncation. Thus the dominant truncation of equation 5 is

$$u_{xx} = 6u^2 \tag{7}$$

Equation 7 is also the Painlevé canonical equation II which is solvable by elliptic functions.[7]

The next simplest irreducible Painlevé equation is

$$u_{xx} = 2u^3 + xu + b \tag{8}$$

Analogously expressing u in equation 8 by equation 4 gives

$$ap(p-1)(x - x^*)^{p-2} = 2a^3(x - x^*)^{3p} + ax(x - x^*)^p + b \tag{9}$$

Balancing the $(x - x^*)^{p-2}$ term on the left with the $(x - x^*)^{3p}$ term on the right gives $p - 2 = 3p$ or $p = -1$. The $(x - x^*)^p$ term on the right of equation 9 cannot be involved in the balancing since $p - 2 \neq p$. This term is therefore a recessive term as is the constant term b of equation 8. Therefore the dominant truncation of equation 8 is

$$u_{xx} = 2u^3 \tag{10}$$

A simple type of second order differential equation reducible to a first order differential equation of Riccati type is the Painlevé canonical equation V, namely

$$u_{xx} = -2uu_x + bu_x + b'u \tag{11}$$

Methods analogous to those used above indicate that p = -1 for equation 11 and that its dominant truncation is

$$u_{xx} = -2uu_x \tag{12}$$

Thus equations 8 and 11 lead to the same value for the exponent p when expanded in the neighborhood of a critical point by using equation 4 but lead to dominant truncations having very different forms. Thus equation 8 and 11 are in different dominant families.

Second order differential equations without movable critical points are also possible which have dominant truncations which are linear combinations of equations of the types 10 and 12. This phenomenon, which can be called hybridization, is possible because equations 10 and 12 lead to the same value of the exponent p when equation 4 is substituted into them. The simplest example of hybridization in the Painlevé canonical equations[7] occurs in Painlevé equation VI, namely

$$u_{xx} = -3uu_x - u^3 + bu_x - bu^2 \tag{13}$$

The dominant truncation of equation 13 is

$$u_{xx} = -3uu_x - u^3 \tag{14}$$

This is a linear combination of equations 10 and 12 with appropriate adjustments of the multiplicative constants.

Another phenomenon is observed in the third and higher irreducible Painlevé equations.[7] Thus the third irreducible Painlevé equation (canonical form[7] XIII) is

$$u_{xx} = (u_x^2/u) + bu^3 + cu^2 + d + (e/u) \tag{15}$$

Substituting equation 4 into equation 10 gives

$$\begin{aligned} ap(p-1)(x - x^*)^{p-2} &= ap^2(x - x^*)^{p-2} + ba^3(x - x^*)^{3p} + ca^2(x - x^*)^{2p} \\ &\quad + d + (e/a)(x - x^*)^{-p} \end{aligned} \tag{16}$$

The $ap^2(x - x^*)^{p-2}$ and $ba^3(x - x^*)^{3p}$ terms on the right of equation 16 are both dominant terms but only the $ba^3(x - x^*)^{3p}$ term can be used to determine the exponent p to be -1. The $ba^3(x - x^*)^{3p}$ term may therefore be considered to be an <u>active</u> dominant term. Similarly $ap^2(x - x^*)^{p-2}$ may be regarded as a <u>passive</u> dominant term. Self-dominant equations having only active dominant terms may be called <u>strongly self-dominant</u> equations. Since passive dominant terms are not found in the evolution equations of interest, only strongly self-dominant equations will be considered in this paper. These will be seen to relate closely to the evolution equations.

III. PAINLEVÉ ANALYSIS OF STRONGLY SELF-DOMINANT EQUATIONS

We will now consider the general features of the so-called Painlevé analysis used to determine whether a given strongly self-dominant differential equation has the Painlevé property. Such equations can be expressed as polynomials of the following type:

$$u_{nx} = g(x,u,u_{1x},\ldots,u_{(n-1)x}) = \sum_i c_i \prod_{j=0}^{n-1} u_{jx}^{q_j} \tag{17}$$

in which n is thus the order of the differential equation. Let us now assign to equation 17 the following integers:

n = order of the equation (order of the derivative u_{nx} on the left-hand side of equation 17 which is the highest order derivative in the equation).

$m_i = \sum_{j=0}^{n-1} jq_j$ (the weighted sums of the derivatives in the terms on the right-hand side of equation 17)

$d_i = \sum_{j=0}^{n-1} q_j$ (degrees of the polynomial terms on the right-hand side of equation 17)

In general $m_i \neq m_k$ and $d_i \neq d_k$. However, initially we shall consider homogeneous equations (17) in which $m_i = m_k$ and $d_i = d_k$ for all values of i and k. For such a homogeneous equation we can assign unique values of m and d. Let us call m and d the co-order and the degree, respectively, of the equation.

Let us now apply Painlevé analysis to equation 17. Express u as the power series in equation 4. Determine the exponent p which balances the terms. By taking appropriate derivatives of equation 4 the following relationship can be seen to hold:

$$p = \frac{m - n}{d - 1} \tag{18}$$

If p is not an integer, then equation 17 has a movable algebraic branch point implying non-Painlevé behavior. We are therefore interested in self-dominant systems of the type represented by equation 17 for which p as determined by equation 18 is a negative integer.

If p (equation 18) is a negative integer, then equation 4 may represent the first term in a Laurent series[8] valid in a deleted neighborhood of a movable pole. In this case a solution of equation 17 is of the following type:

$$u = (x - x^*)^p \sum_{k=0}^{\infty} a_k (x - x^*)^k \tag{19}$$

where $x - x^* \neq 0$. In this case the position x^* of the singular value of x corresponds to one of the n integration constants. If n-1 of the coefficients a_k are also arbitrary, the n integration constants of equation 17 are then accounted for and equation 19 represents the solution of equation 17 in the deleted neighborhood of the singularity x^*. The powers of x at which these arbitrary constants enter are called the resonances and will be designated as $r_1, r_2, \ldots, r_n$ so that $r_i < r_k$ for $i<k$.

In order to find the resonances the following equation for u is substituted into equation 17:

$$u = a(x - x^*)^p + b(x - x^*)^{p+r} \tag{20}$$

The coefficient a is obtained by equating the coefficients of the $(x - x^*)^{p-n}$ terms which are the leading terms in the neighborhood of x^*. For a homogeneous equation 17 the coefficient a is uniquely determined. After determining a then the coefficients of the next higher powers $(x - x^*)^{p+r-n}$ are equated in order to determine the resonances. In this way the resulting equations for the resonances to leading order in b reduce to

$$Q(r) b (x - x^*)^q = 0 \qquad q \geq p+r-n \tag{21}$$

in which Q(r) is a polynomial in r of degree n. The roots of Q(r) determine the resonances since Q(r) = 0 corresponds to the "indicial equation" used to solve a linear ordinary differential equation near a regular singular point.[9]

Let us now consider some features of this indicial polynomial Q(r). Because of the rules for differentiation, the left-hand side of equation 17 will generate the n-th degree polynomial L(r) of the following type:

$$L(r) = (r + p)(r + p - 1) \ldots (r + p - n) \tag{22}$$

Since p is a negative integer, L(r) is not divisible by r+1 and all of its roots are real positive integers. However, the polynomial Q(r) must be divisible by r+1 reflecting the arbitrariness of the singularity $x=x^*$ corresponding to one of the n integration constants. This leads to an automatic root of -1 for Q(r). Therefore substitution of equation 20 into the right-hand side of equation 17 must generate a polynomial R(r) so that the difference L(r) - R(r) is divisible by r+1 and is factorable into linear factors so that all of its roots are real integers. However, the degree of R(r) is m<n so that the terms in L(R) of the type $h_k r^{n-k}$ in which n-k>m will be the same as the corresponding terms of the indicial polynomial Q(r).

Arguments based on the relationships between the coefficients of the highest degree terms of polynomials and the sums of powers of their roots[10] indicate that the degree of R(r), which corresponds to the co-order m of the original differential equation (17), must be high enough so that the difference L(r) - R(r) with L(r) defined by equation 22 becomes divisible by r+1 while remaining factorable into linear factors even though L(r) itself is not divisible by r+1. Otherwise the corresponding differential equation (17) will not have the Painlevé property. For this reason only differential equations (17) having p = -1 or p = -2 can be candidates for equations having the Painlevé property.

The next step is to find the roots of Q(r). If all of the roots of Q(r) other than the automatic -1 roots are real integers with at least one root greater than -1, then the system can be free from algebraic branch points. The corresponding homogeneous differential equation (17) is a possible candidate for a system having the Painlevé property. The complete Painlevé analysis requires additional steps involving determination of the integration constants.[5] These additional steps are sufficiently more complicated and tedious so that they cannot readily be applied to the diverse variety of systems considered in this paper. We shall therefore limit the discussion in this paper to the identification of the types of differential equations (17) which can lead to the integral resonances required for the Painlevé property.

IV. RESULTS

Table 1 lists all of the possible types of strongly self-dominant homogeneous differential equations of order $\leq$ 4 which have the negative integral balancing exponents p (equation 18) required for the Painlevé property. Of these sixteen equation types, nine are shown by the methods outlined above to have the real integral resonances required for the Painlevé property. These nine equations are all members of the three Painlevé chains described in Table 2. In this context a Painlevé chain consists of a chain of differential equations of the following type:

$$u_{nx} = \left(\frac{\partial}{\partial x}\right)^m [u^{(m-n+p)/p}];\ m = 1,2,... \tag{23}$$

The Painlevé chains may be characterized by the fraction (n-m)/(d-1) which by equation 18 is the negative of the exponent p. The three Painlevé chains depicted in Table 2 are the only possible Painlevé chains for which n-m $\leq$ 2 and therefore for reasons outlined in the previous section are the only possible Painlevé chains giving the integral resonances required for the Painlevé property. Note also that each member of a Painlevé chain has all of the resonances of the previous member plus one additional resonance. In ascending a Painlevé chain to higher orders through the successive differentiations implied by equation 23, points may be reached where the indicial polynomial Q(r) has a multiple root (i.e., order 4 for the 2/2 chain and order 5 for the 2/1 chain) and a point is reached where the right-hand side of the differential equation splits into more than one term. The algorithm for determining the resonances is independent of the parameter k (see Table 1) since products of the type ka^{d-1} (a from equation 4 for u) are constant. However, in equations having multiple terms on the right-hand side, the positions of the resonances depend upon the ratios of the coefficients of these terms. However, the process of obtaining the members of a Painlevé chain through equation 23 and the implied successive differentiations suggest ratios between the coefficients of the terms on the right-hand side as given in Table 1 and 2 which may have special significance.

Table 2 also indicates the relationships of the Painlevé chains to the evolution equations. Each of the three Painlevé chains contains at least one of the evolution equations. The 2/1 chain is the most important one in this connection since it contains both the Korteweg-de Vries and Boussinesq equations. The higher order

TABLE 1
SELF-DOMINANT HOMOGENEOUS DIFFERENTIAL EQUATIONS OF ORDER ≤ 4
WITH NEGATIVE INTEGRAL BALANCING EXPONENT p

Equation Type	Order n	Co-order m	Degree d	Exponent p	Fraction (n-m)/(d-1)	Resonances	Equation Type[a]
$u_{2x} = ku^3$	2	0	3	-1	2/2	-1,+4	DT of Painlevé II
$u_{2x} = kuu_x$	2	1	2	-1	1/1	-1/+2	Burgers
$u_{2x} = ku^2$	2	0	2	-2	2/1	-1,+6	DT of Painlevé I
$u_{3x} = ku^4$	3	0	4	-1	3/3	Complex	
$u_{3x} = ku^2u_x$	3	1	3	-1	2/2	-1,+3,+4	Modified KdV
$u_{3x} = k(u_x^2 + uu_{2x})$	3	2	2	-1	1/1	-1,+2+3	
$u_{3x} = kuu_x$	3	1	2	-2	2/1	-1,+4,+6	Korteweg-de Vries
$u_{3x} = ku^2$	3	0	2	-3	3/1	Complex	
$u_{4x} = ku^5$	4	0	5	-1	4/4	Complex	
$u_{4x} = ku^3u_x$	4	1	4	-1	3/3	Complex	
$u_{4x} = k(2uu_x^2+u^2u_{2x})$	4	2	3	-1	2/2	-1,+3,+4,+4	
$u_{4x} = k(uu_{3x}+3u_xu_{2x})$	4	3	2	-1	1/1	-1,+2,+3,+4	
$u_{4x} = ku^3$	4	0	3	-2	4/2	Complex	
$u_{4x} = k(u_x^2+uu_{2x})$	4	2	2	-2	2/1	-1,+4,+5,+6	Boussinesq
$u_{4x} = kuu_x$	4	1	2	-3	3/1	Complex	
$u_{4x} = ku^2$	4	0	2	-4	4/1	Complex	

a) DT = dominant truncation

TABLE 2
PAINLEVÉ CHAINS OF SELF-DOMINANT HOMOGENEOUS DIFFERENTIAL EQUATIONS WITH INTEGRAL RESONANCES

<u>2/2 Chain</u>

$u_{2x} = ku^3 \longrightarrow$	$u_{3x} = ku^2u_x \longrightarrow$	$u_{4x} = k(2uu_x^2 + u^2u_{2x}) \longrightarrow$	$u_{5x} = k(2u_x^3 + 6uu_xu_{2x} + u^2u_{2x})$
(+4)	(+3, +4)	(+3, +4, +4)	(+3, +4, +4, +5)
Dominant Truncation of			
Painlevé II	Modified kdV		

<u>1/1 Chain</u>

$u_{2x} = kuu_x \longrightarrow$	$u_{3x} = k(uu_{2x} + u_x^2) \longrightarrow$	$u_{4x} = k(uu_{3x} + 3u_xu_{2x}) \longrightarrow$	$u_{5x} = k(uu_{4x} + 4u_xu_{3x} + 3u_{2x}^2)$
(+2)	(+2, +3)	(+2, +3, +4)	(+2, +3, +4, +5)
Burgers			

<u>2/1 Chain</u>

$u_{2x} = ku^2 \longrightarrow$	$u_{3x} = kuu_x \longrightarrow$	$u_{4x} = k(u_x^2 + uu_{2x}) \longrightarrow$	$u_{5x} = k(3u_xu_{2x} + uu_{3x})$
(+6)	(+4, +6)	(+4, +5, +6)	(+4, +5, +6, +6)
Dominant Truncation of			
Painlevé I	Korteweg-de Vries	Boussinesq	

equations in the three Painlevé chains of Table 3 are interesting candidates for detailed future study, since some of them are possibilities for new equations solvable by the inverse scattering transform or related methods.

Note from Tables 1 and 2 that the 2/2 and 1/1 chains have the same exponent p, namely -1. The homogeneous equations of a given order in these chains can be mixed to give a hybrid equation which is still self-dominant. Similar mixing of homogeneous differential equations of a given order and fractions 4/2 and 2/1 (i.e., $p = -2$) can also give hybrid self-dominant equations; this latter type of mixing is important in the study of fifth-order evolution equations as discussed below. Note that mixing a 4/2 equation with a 2/1 equation of the same order can give a hybrid equation with the integral resonances required for the Painlevé property even though a pure 4/2 equation cannot have the Painlevé property since it has complex resonances rather than integral resonances.

Let us consider hybridization of the third-order equations $u_{3x} = k(u_x^2 + uu_{2x})$ (fraction 1/1) and $u_{3x} = k'u^2u_x$ (fraction 2/2); the latter is the dominant truncation of the modified (cubic) KdV equation. If the 1/1 coefficient ratio of the u_x^2 and uu_{2x} terms is preserved, the resulting hybrid can be expressed in the following form:

$$u_{3x} = u_x^2 + uu_{2x} + hu^2u_x \tag{24}$$

Balancing the $(x - x^*)^{-4}$ terms according the Painlevé procedure gives the following quadratic equation for a:

$$a^2 - \frac{3a}{h} - \frac{6}{h} = 0 \tag{25}$$

Therefore

$$a = \frac{3}{2h} \pm \frac{1}{2h}\sqrt{9 + 24h} \tag{26}$$

Thus for any value of $h > -3/8$, a has two values indicating two solution branches. Such multiple solution branches are a characteristic of hybrid equations.

Table 3 illustrates the Painlevé analysis for the hybrid third-order differential equation 24 using as examples several values of h leading to rational values of a. In three of the four cases one of the two solution branches has a "good" set of resonances for the Painlevé property (resonances at -1 and two other distinct integers including at least one positive integer) > -1 and the other branch has

TABLE 3

PAINLEVÉ ANALYSIS OF THE THIRD ORDER HYBRID EQUATION

$$u_{3x} = u_x^2 + uu_{xx} + hu^2u_x$$

h	a	resonances	Properties
2/3	6	-1, +3, +10	good
2/3	-3/2	-1, +3, +5/2	bad: non-integral root
3	2	-1, +3, +6	good
3	-1	-1, +3, +3	bad: double root
9	1	-1, +3, +5	good
9	-2/3	-1, +3, +10/3	bad: non-integral root
-1/3	-3	-1, +1, +3	good
-1/3	-6	-2, -1, +3	satisfactory

a "flaw" in its set of resonances (non-integral resonance or a double root). Note, however, that regardless of the values of h and a, two of the three resonances in equation 24 appear at -1 and +3 which are the two resonances possessed by both components of the hybrid, namely $u_{3x} = k(u_x^2 + uu_{2x})$.

Table 4 summarizes the Painlevé analysis for the three homogeneous fifth-order differential equations which are members of the three Painlevé chains in Table 2 as well as three hybrid fifth-order differential equations which have been studied as evolution equations. All three of the hybrid systems are mixtures of the 2/1 quadratic and 4/2 cubic fifth-order differential equations. The following points about these fifth-order hybrid differential equations are of interest:

(1) In contrast to the third order differential equations of Table 3 the ratios between the coefficients of the u_xu_{2x} and uu_{3x} terms of the 2/1 components (designated as r_{12}/r_{03}) are different for the hybrid systems than for the pure homogeneous fifth-order 2/1 equation generated by the successive differentiations implied by equation 23. Thus for the homogeneous systems r_{12}/r_{03} is 3 whereas for the hybrid systems r_{12}/r_{03} is 1, 5/2, and 2 for the Caudrey-Dodd-Gibbon, the Kuperschmidt, and the higher order Korteweg-de Vries equations, respectively. The hybrid systems may therefore be viewed as being generated from the pure 2/1 system by perturbing the ratio r_{12}/r_{03} from that generated for the pure system by differentiation (namely 3) and then mixing in enough of the 4/2 equation (i.e., $u_{5x} = k'u^2u_x$) to restore the integral resonances required for the Painlevé property.

(2) For reasons noted above the fifth-order 2/1 + 4/2 hybrid equations have two solution branches arising from the two roots of a quadratic equation analogous to equation 25. In all three cases one solution branch has all integers greater than the mandatory -1 resonance whereas the other solution branch has one resonance below -1 in addition to three distinct integral resonances greater than -1.

TABLE 4
SOME FIFTH-ORDER STRONGLY SELF-DOMINANT DIFFERENTIAL EQUATIONS WITH INTEGRAL RESONANCES

Equation Type	Co-order m	Degree d	Exponent p	Fraction (n-m)/(d-1)	Resonances[a]	Reference
A). Homogeneous Equations (n=5)						
$u_{5x} = k(2u_x^3 + 6uu_xu_{2x} + u^2u_{3x})$	3	3	-1	2/2	-1,+3,+4,+4,+5	
$u_{5x} = k(uu_{4x} + 4u_xu_{3x} + 3u_{2x}^2)$	4	2	-1	1/1	-1,+2,+3,+4,+5	
$u_{5x} = k(3u_xu_{2x} + uu_{3x})$	3	2	-2	2/1	-1,+4,+5,+6,+6	
B). Hybrid Equations (n=5)						
$u_{5x} = -30u_xu_{2x} - 30uu_{3x} - 180u^2u_x$	3,1	2,3	-2	2/1+4/2	-1,+2,+3,+6,+10 -2,-1,+5,+6,+12	b
$u_{5x} = -25u_xu_{2x} - 10uu_{3x} - 20u^2u_x$	3,1	2,3	-2	2/1+4/2	-1,+3,+5,+6,+7 -7,-1,+6,+10,+12	c
$u_{5x} = -20u_xu_{2x} - 10uu_{3x} - 30u^2u_x$	3,1	2,3	-2	2/1+4/2	-1,+2,+5,+6,+8 -3,-1,+6,+8,+10	d

(a) The resonances of both solution branches are listed for the hybrid systems.

(b) The Caudrey-Dodd-Gibbon equation: P.J. Caudrey, R.K. Dodd, and J.B. Gibbon, Proc. Roy. Soc. London, A351, 407 (1976).

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V. SUMMARY

This paper demonstrates a simple way of classifying higher order differential equations based on the requirements of the Painlevé property related to the solubility of the equation by inverse scattering transform methods.[1,2,3] As expected the known evolution equations such as the Korteweg-de Vries, Burger, Boussinesq, and Caudrey-Dodd-Gibbon equations occupy prominent positions in this classification scheme. This classification scheme also identifies new potential candidates for higher order differential equations with the Painlevé property and possibly soluble by inverse scattering transform methods. A major objective of this paper is to stimulate further work which hopefully will relate the ideas in this paper to such important aspects in the study of evolution equations as the generation of Lax pairs,[11,12] conservation laws,[1,2] Bäcklund transformations,[12,13,14,15,16] recursion relations,[6,12] Schwarzian derivatives,[12,14,15,16,17] and prolongation structures[18,19] as well as details of the inverse scattering transform procedure.[1,2,3]

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